Linden 76

Soil Science 77 Jan June 1954

Philip

AN INFILTRATION EQUATION WITH PHYSICAL SIGNIFICANCE

J. R. PHILIP

Commonwealth Scientific and Industrial Research Organization, Australia[1]

Recieved for publication May 7, 1953

Hydrologists use the term "infiltration" to describe the entry into the soil of water available at the soil surface. It is important to determine the functional relationship between the total infiltration (i) and the time for which the infiltration has proceeded (t).

Kostiakov (8) appears to have first suggested the infiltration equation

$$i = Kt^{\alpha} \tag{1}$$

which has been used widely. Another common equation

$$f_t = f_{\infty} + (f_0 - f_{\infty})e^{-\beta t} \tag{2}$$

(where f_t is the infiltration rate at time t, and f_0 the initial, and f_{∞} the final rate) was first proposed by Gardner and Widstoe (3) and later by Horton (5).

These two equations are empirical (and only moderately successful) attempts to fit experimental data. Little significance can be attributed to the parameters in them.

An endeavor to develop the infiltration equation in terms of the basic physical properties of the system might be expected to stem from the general equation of flow for a liquid in a homogeneous porous medium of stable structure,

$$\frac{\partial}{\partial t}(\rho m) = \nabla \cdot \left(\frac{\rho}{\mu} k \nabla \Phi\right) \tag{3}$$

where m is the liquid content of the medium on a volumetric basis, k its capillary conductivity at liquid content m, ρ is the liquid density, μ the liquid viscosity, and Φ the total potential (considered here to comprise gravity and capillary potential). For a one-dimensional system of vertical flow, equation (3) reduces to

$$\frac{\partial}{\partial t}(\rho m) = \frac{\partial}{\partial z}\left(\frac{\rho}{\mu} k \frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\rho}{\mu} kg\right) \tag{4}$$

where z is the vertical ordinate and Ψ is the capillary potential. Equations (3) and (4) are more general forms of equations given by Klute (6, 7).

Equation (4) describes the infiltration of a liquid under any given conditions of initial liquid content in the profile. In view of the dependence of k and Ψ on m, solutions of this equation will be found only for particular numerical cases.

[1] Contribution from the Regional Pastoral Laboratory, Deniliquin, New South Wales.

DERIVATION OF APPROXIMATE INFILTRATION EQUATION

Available experimental data (1, 2, 4, 10, 11) indicate that the following equalities hold approximately:

$$\int_{-x}^{0} m\,dz = \bar{m}x \tag{5}$$

$$\int_{-x}^{0} \frac{v\,dz}{k} = \frac{Vx}{k_h} \tag{6}$$

where x cm. is the depth of the wetting front, v is the flow velocity at depth $-z$, V the value of v at $z = 0$, and $\bar{m}$ and k_h are both taken as constant and independent of x. The analysis which the assumption of (5) and (6) makes possible certainly evades the question of the exact status of the "wetting front" but appears to give useful results nevertheless.

Consider the downward entry of a liquid into a soil with initial liquid content m_0. Then the difference in total potential from $z = 0$ to $z = -x$ expressed in centimeters of liquid is equal to $-(P + x + H)$ where H cm. is the depth of liquid above the soil and $-P$ cm. of liquid is the capillary potential at the wetting front. The use of Darcy's law enables this difference in potential to be expressed also as a function of flow velocity, namely

$$-\int_{-x}^{0} \frac{\mu v\,dz}{k}, \qquad \text{or [using (6) and taking } \mu \text{ constant]} \qquad -\frac{\mu V x}{k_h}.$$

Therefore $\dfrac{\mu V x}{k_h} = P + x + H$ which reduces to $V = \dfrac{k_h}{\mu}\left[1 + \dfrac{(P + H)}{x}\right]$ (7)

But $V = \dfrac{di}{dt}$ and $x = \dfrac{i}{(m - m_0)}$, so that (7) may be rewritten

$$\frac{di}{dt} = \frac{k_h}{\mu}\left(1 + \frac{(m - m_0)(P + H)}{i}\right) \tag{8}$$

The particular integral of (8) which vanishes for $t = 0$ is

$$t = \frac{\mu}{k_h}\left[i - (P + H)(m - m_0)\log\left(1 - \frac{i}{(P + H)(m - m_0)}\right)\right] \tag{9}$$

Now, from the elementary theory of capillarity

$$P = \frac{2\gamma\cos\theta}{\rho g r} \tag{10}$$

where γ is the surface tension of the liquid, θ is the angle of contact of the liquid-air-soil system under conditions of liquid advance, and r cm. is the radius of the largest hypothetical circular "pore" containing liquid at the wetted front. Note that θ and r are assumed independent of time. Substitution of (10) in (9) gives the approximate infiltration equation in its most general form

$$t = \frac{\mu}{k_h}\left[i - \left(\frac{2\gamma\cos\theta}{\rho g r} + H\right)(\bar{m} - m_0)\log\left(1 + \frac{i}{\left(\frac{2\gamma\cos\theta}{\rho g r} + H\right)(\bar{m} - m_0)}\right)\right] \quad (11)$$

Green and Ampt (4) derived an equation of the form

$$t = Y[i - Z\log(1 + i/Z)] \quad (12)$$

but they gave little explanation of the constant Z.

DISCUSSION

Exploration of the implications of this analysis, a few of which are discussed here, may prove profitable.

Observations (9) that infiltration appears to be independent of the depth of water over the soil can be satisfactorily explained by equation (*9*). The value of P for water ranges from 80 cm. for soils of coarse texture to 140 cm. for clay (12). The depth of water over the soil is commonly only a few centimeters, so the effect on z of the water depth is small. Thus, for $P = 100$ cm., variation of H from 1 to 5 cm. produces only a 4 per cent change in z.

Equation (*8*) implies that the infiltration rate is primarily dependent on i. This provides some formal justification for the technique of "time-condensation," which hydrologists (14) have developed (apparently intuitively) to enable the use of an infiltration equation derived for the case of continuous surplus water at the surface in the more general problem where water may be made available at rates less than the infiltration rate.

The initial moisture content of a soil exerts a major influence on its infiltration characteristics. Emprical studies (13, 15) have established that high infiltration rates are associated with low moisture content, and *vice versa*. Equation (*9*) supplies the functional relationship between moisture content and infiltration. For m and k_h independent of m_0, it is clear from equation (*8*) that di/dt (and therefore i for any given t) decreases as m_0 increases. Values of m and k_h will vary slightly with m_0, which will modify this simple relationship to some degree. Colman and Bodman (2) have given some attention to these factors.

CONDITIONS OF HYDRAULIC SIMILITUDE

Φ in equation (*3*) may be written

$$\Phi = L + A\,\frac{\gamma\cos\theta}{\rho} \quad (13)$$

the first term denoting the gravity potential, and the second the capillary potential.

Let suffixes 1 and 2 denote values of the symbols for the movement of liquids 1 and 2 through the same porous medium. Then, for $\frac{\Phi_1}{\Phi_2}$ constant, and

$$L_1 \Big/ \frac{A_1 \gamma_1 \cos \theta_1}{\rho_1} = L_2 \Big/ \frac{A_2 \gamma_2 \cos \theta_2}{\rho_2},$$

which are conditions of similitude,

$$\frac{\Phi_1}{\Phi_2} = \frac{L_1}{L_2} = \frac{\gamma_1 \cos \theta_1 \rho_2}{\gamma_2 \cos \theta_2 \rho_1} \tag{14}$$

It is clear that L_1/L_2 represents the ratio of the respective length scales. Now for ρ independent of time and position, (*3*) may be rewritten

$$\frac{\partial m}{\partial t} = \nabla \cdot \left(\frac{k}{\mu} \nabla \Phi \right) \tag{15}$$

If T_1 and T_2 denote the respective time scales

$$\frac{T_1}{T_2} = \left(\frac{\partial m}{\partial t}\right)_2 \Big/ \left(\frac{\partial m}{\partial t}\right)_1 = \frac{L_2^{-2}\Phi_2/\mu_2}{L_1^{-2}\Phi_1/\mu_1} = \frac{L_1 \mu_1}{L_2 \mu_2} = \frac{\mu_1 \gamma_1 \cos \theta_1 \rho_2}{\mu_2 \gamma_2 \cos \theta_2 \rho_1} \tag{16}$$

Therefore a model of unsaturated water movement which is in hydraulic similitude with the prototype may be made with the same soil and a liquid other than water. The scale ratios will be given by equations (*14*) and (*16*).

It can be shown simply that these conditions of similitude also hold for the approximate analysis which led to equation (*11*).

Suitable selection of an infiltering liquid to compress time and length scales may make laboratory investigation of large-scale phenomena of infiltration and unsaturated flow practical and economical. Exploratory work with this end in view is proposed at this laboratory.

It is emphasized that the analysis developed here is for a homogeneous soil of stable structure. This simple case must, of course, be studied before progress is likely on those of greater complexity.

The terminology of this communication follows the recommendations of a recent committee (16).

SUMMARY

An approximate equation is developed which relates infiltration to the physical determinants of the system. This equation gives quantitative explanation of aspects of infiltration.

The approach of Klute is generalized and used to obtain the condition for hydraulic similitude between prototype phenomena in unsaturated movement of water in the liquid phase and models employing the same medium but a liquid other than water.

REFERENCES

(1) Bodman, G. B., and Colman, E. A. 1944 Moisture and energy conditions during downward entry of water into soils. *Soil Sci. Soc. Amer. Proc.* (1943) 8: 116–122.

(2) Colman, E. A., and Bod
downward entry of
(1944) 9: 3–11.

(3) Gardner, W., and Wids
215–232.

(4) Green, W. H., and Amp
water through soils.

(5) Horton, R. E. 1940
capacity. *Soil Sci. S*

(6) Klute, A. 1952 Numer
materials. *Soil Sci.*

(7) Klute, A. 1952 Some
Soil Sci. Soc. Amer.

(8) Kostiakov, A. N. 1932
soils and on the ne
purposes of amelior
17–21.

(9) Lewis, M. R. and Pow
Soil Sci. Soc. Amer.

(10) Miller, R. D. 1951 T
changing systems. *S*

(11) Miller, R. D. and Ric
soils. *Soil Sci. Soc.*

(12) Moore, R. E. 1939 W
383–426.

(13) Neal, J. H. 1938 Effec
and soil erosion. Un

(14) Sherman, L. K. 1943
and Holtan and of S

(15) Tisdall, A. L. 1951
Australian J. Agr.

(16) Richards, L. A. 1952
committee on termi
Proc. (1951) 16: 85–

(2) Colman, E. A., and Bodman, G. B. 1945 Moisture and energy conditions during downward entry of water into moist and layered soils. *Soil Sci. soc. Amer. Proc.* (1944) 9: 3–11.

(3) Gardner, W., and Widstoe, J. A. 1921 Movement of soil moisture. *Soil Sci.* 11: 215–232.

(4) Green, W. H., and Ampt, G. A. 1911 Studies on soil physics: I. Flow of Air and water through soils. *J. Agr. Research* 4: 1–24.

(5) Horton, R. E. 1940 Approach toward a physical interpretation of infiltration capacity. *Soil Sci. Soc. Amer. Proc.* (1939) 5: 399–417.

(6) Klute, A 1952 Numerical method for solving the flow equation for water in porous materials. *Soil Sci.* 73: 105–116.

(7) Klute, A. 1952 Some theoretical aspects of the flow of water in unsaturated soils. *Soil Sci. Soc. Amer. Proc.* 16 (1951): 144–148.

(8) Kostiakov, A. N. 1932 On the dynamics of the coefficient of water-percolation in soils and on the necessity for studying it from a dynamic point of view for purposes of amelioration. *Trans. 6th Com. Intern. Soc. Soil Sci. Russian* Part A: 17–21.

(9) Lewis, M. R. and Powers, W. L. 1938 Study of factors affecting infiltration. *Soil Sci. Soc. Amer. Proc.* (1937) 334–338.

(10) Miller, R. D. 1951 Technique for measuring soil-moisture tensions in rapidly changing systems. *Soil Sci.* 72: 291–301.

(11) Miller, R. D. and Richard, F. 1952 Hydraulic gradients during infiltration in soils. *Soil Sci. Soc. Amer. Proc.* (1951) 16: 33–38.

(12) Moore, R. E. 1939 Water conduction from shallow water tables. *Hilgardia* 12: 383–426.

(13) Neal, J. H. 1938 Effect of the degree of slope and rainfall characteristics on runoff and soil erosion. Univ. Missouri Research Bull. 280.

(14) Sherman, L. K. 1943 Comparison of f-curves derived by the methods of Sharp and Holtan and of Sherman and Mayer. *Trans. Am. Geophys. Union* 24: 465–467.

(15) Tisdall, A. L. 1951 Antecedent soil moisture and its relation to infiltration. *Australian J. Agr. Research* 2: 342–348.

(16) Richards, L. A. 1952 Report of the subcommittee on permeability and infiltration, committee on terminology, Soil Science Society of America. *Soil Sci. Soc. Amer. Proc.* (1951) 16: 85–88.