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THE THEORY OF INFILTRATION: 4. SORPTIVITY AND ALGEBRAIC INFILTRATION EQUATIONS

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Many situations in applied hydrology require that the dynamics of infiltration be characterized by a small number of parameters. These parameters are most appropriately the coefficients of an algebraic equation representing the variation of i or v_0 with t.[2]

Parts 1 and 2 of this series (12, 13) provided a detailed analysis of infiltration and part 3 (14) a general discussion of the physical significance of the analytical results. In the present paper we use the preceding work as the basis for a study of the available (generally empirical) algebraic infiltration equations.

This study is facilitated if we first give some attention to a new physical property of porous media which enters the subsequent developments here and which we shall have occasion to use also in later papers of this series.

SORPTIVITY

It is evident from equation (*38*) of part 1 (12) that the most important single quantity governing infiltration at small t is $\int_\varphi$. It is equally apparent that, for t sufficiently large, the dominant quantity is K_0 [cf. equation (*27*) of part 2 (13)]. The physical significance of K_0 is clear, but additional remarks on $\int_\varphi$ are required. When $\int_\varphi$ is discussed alone, rather than, for example, as a member of the sequence $\int_\varphi$, $\int_\chi$, etc. it will be denoted by the more convenient symbol S. For any medium subject to conditions (*20*) of part 1 (12) S is clearly a function of θ_0 and θ_n.

Terminology

Since S is a measure of the capillary uptake or removal of water, it is essentially a property of the medium with some resemblance to permeability. "Absorptivity" (9) would be a suitable name for S quite comparable to "permeability" or "conductivity." Since, however, a term embracing both absorption and desorption is desired, it is proposed to use the more general "sorptivity." Although this rather extends the meaning of the "sorption" of McBain (8), the extension seems warranted and not ambiguous.

[1] Contribution from the Division of Plant Industry, Deniliquin, New South Wales.

[2] Definitions of the more important symbols used in this series are given in an earlier paper (12).

Units and dimensions

Now for absorption into a horizontal column [equatiod (*9*) subject to conditions (*10*) of part 1 (12)]

$$v_0 = \frac{S}{2} t^{-1/2} \quad (1)$$

so that when v_0 is in cm. sec.$^{-1}$ and t in sec., the *practical unit* of S is cm. sec.$^{-1/2}$.

We write the equation for horizontal movement in a fundamental form appropriate to liquids which do not react with the medium and have contact angle H at the liquid-solid-air line of contact:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\sigma \cos H}{\mu} \mathfrak{D} \frac{\partial \theta}{\partial x}\right) \quad (2)$$

where σ and μ are respectively the surface tension and viscosity of the liquid. The new diffusivity, $\mathfrak{D}$, is a function of the medium geometry, but is independent of the properties of the liquid. It might be termed the *intrinsic diffusivity*, though it is notable that its dimensions are those of (length) in contrast to the dimensions of classical diffusivity, (length)^2(time)$^{-1}$.

$\mathfrak{D}$ is, of course, connected with D through the relation

$$\mathfrak{D} = \frac{\mu D}{\sigma \cos H} \quad (3)$$

The fundamental form of solution (*1*) is now found to be

$$v_0 = \frac{\mathcal{S}}{2}\left(\frac{\sigma \cos H}{\mu t}\right)^{1/2} \quad (4)$$

where the new sorptivity $\mathcal{S}$ is the *intrinsic sorptivity* of the medium, and, like the intrinsic permeability (*15*), is essentially an expression of the geometry of the medium.

$\mathcal{S}$ is related to S by the equation

$$\mathcal{S} = \left(\frac{\mu}{\sigma \cos H}\right)^{1/2} S \quad (5)$$

Dimensional analysis shows that the dimensions of $\mathcal{S}$ are (length)$^{1/2}$. It is interesting to compare this result with the dimensions of intrinsic permeability, (length)^2.

The author is grateful to a *Soil Science* reviewer for drawing his attention to a paper by Swartzendruber *et al.* (16) who previously gave some attention to these matters. The "square root of time proportionality constant" or "capillary absorption coefficient" C of Swartzendruber *et al.* is equal to $(60)^{1/2}$ S in the present notation. These authors used an idealized capillary tube model of the soil which led them to the conclusion that their C varied as the square root of the "capillary radius," while the hydraulic conductivity varied as the square of the same quantity.

It will be noted that the present developments provide a very wide generalization of this result of Swartzendruber *et al.* We have shown that (with their "capillary radius" replaced by a length characteristic of the medium) this is valid for perfectly general medium geometry and, further, for any (non-reactive) sorbate.

ALGEBRAIC INFILTRATION EQUATIONS

We now proceed to relate the various available infiltration equations to the analysis developed in this series. Finally, the goodness of fit of each equation to the analytical result will be tested.

The Horton equation

Horton (4) proposed the equation

$$v_0 = v_f + (v_i - v_f)e^{-\beta t} \tag{6}$$

where v_i is the presumed "initial" value of v_0 and v_f is the presumed "final" value. β is a constant. Gardner and Widtsoe (3) previously proposed a similar equation for the time-dependence of the advance of the "wet front." The two equations are equivalent if the mean moisture content behind the "wet front" is assumed constant (10).

Relating equation (*6*) to the present analysis,

$$v_f = K_0 \tag{7}$$

and v_i is indefinitely great. No physical meaning can be assigned to β.

The integral form of equation (*6*) is:

$$i = v_f t + \frac{1}{\beta}(v_i - v_f)(1 - e^{-\beta t}) \tag{8}$$

The main point in favor of this formulation is that $\lim_{t \to \infty} v_0$ is non-zero. Disadvantages include the fact that it is incapable of adequately representing the very rapid decrease of v_0 from very high values at small t, and the need for three parameters. One might expect that, for a particular soil, the longer the time range, the better this equation would describe infiltration.

The Kostiakov equation

Kostiakov (6) used an equation expressing v_0 as a (negative) power function of t. It is convenient here to write this equation in the form

$$v_0 = \alpha\kappa t^{\alpha-1} \tag{9}$$

where α and κ are parameters. The integral form of equation (*9*) is

$$i = \kappa t^{\alpha} \tag{10}$$

and was developed independently by Lewis (7) and has been used by others, for example, (2) and (17).

Relating these equations to the present analysis we see that

$$\left.\begin{array}{l}\text{in the limit as } t \to 0,\ \alpha = \tfrac{1}{2},\ \kappa = S \\ \text{in the limit as } t \to \infty,\ \alpha = 1,\ \kappa = K_0\end{array}\right\} \tag{11}$$

Since the parameters in equation (*9*) must be constant for the equation to be useful, it is clear that its scope is rather limited. The actual value of α obtained when the equation is fitted to data will clearly depend on the range of t; also, the physical significance of κ depends directly on α and therefore indirectly on the range of t. For a common time range, α will be closer to $\frac{1}{2}$ for fine-textured and initially dry soils in which the capillary potential gradients tend to be relatively more important than the gravitational potential gradients; for wet and sandy soils in which gravity becomes important more rapidly α will be closer to 1. Again the physical significance of κ is obscure and will vary with the α-value.

(We here refer only to homogeneous soils. Obviously if an impermeable horizon underlies a more permeable superficial horizon, α may fall well below the limit of $\frac{1}{2}$ implied in this discussion.)

Despite these difficulties, the Kostiakov equation has the advantage of simplicity and does describe infiltration at the lower end of the time scale quite well. Since for $\alpha < 1$ (which is generally the case) $\lim_{t\to\infty} v_0 = 0$, one expects that equation (*9*), will become less accurate as t increases.

Kostiakov met this limitation by stipulating (in the present symbols) that equation (*9*) holds only for $v_0 > K_0$ and that for $t > (\alpha\kappa/K_0)^{1/1-\alpha}$, $v_0 = K_0$. This procedure presupposes a knowledge of K_0 and is therefore not followed in the subsequent test of this equation.

A recent equation based on a physical model

Recently the writer (10) proposed[3] a simplified physical model of infiltration which led to the equation

$$t = Y[i - Z \log (1 + i/Z)] \qquad (12)$$

where Y and Z are constants which we discuss further. Van Duin (1) has subsequently worked with the same equation.

When this model was reviewed in the light of the present analysis, it was found that equation (*12*) would hold exactly for a soil for which the diffusivity function D may be represented by the limit as $\epsilon \to 0$ of the function $\Delta(\theta)$ defined by

$$\left.\begin{aligned} \Delta(\theta) &= S^2(\theta_0 - \theta_n)/2\epsilon; \; \theta_0 > \theta \geq (\theta_0 - \epsilon). \\ \Delta(\theta) &= 0; \qquad\qquad (\theta_0 - \epsilon) > \theta > \theta_n . \end{aligned}\right\} \qquad (13)$$

that is,

$$D = S^2(\theta_0 - \theta_n)\, \delta(\theta')/2 \qquad (14)$$

where δ is the Dirac delta function (5) and $\theta' = (\theta_0 - \theta)$.

Relating equation (*12*) to this limiting case, we find that coefficients Y and Z become:

$$Y = 1/K_0 \; ; \; Z = S^2/2K_0 \qquad (15)$$

[3] Certain misprints in reference (10) are confusing. The necessary corrigenda follow: p. 154, below equation (7) read everywhere "$\bar{m}$" for "m"; p. 155, "z" and "m" both occur twice and each time "z" should read "Z" and "m" should read "$\bar{m}$".

In the case where the medium is perfectly general, Y and Z may be evaluated by expanding the right-hand side of equation *(12)* as a power series in i, and then forming from it the binomial series for $t^{1/2}$ as a power series in i. These series are then employed in equation *(40)* of part 1 (12) and the coefficients of i (unity) and i^2 (zero) equated on both sides. This gives

$$Y = \frac{2}{3\left[K_n + \int_\chi\right]}; \; Z = \frac{S^2}{3\left[K_n + \int_\chi\right]} \tag{16}$$

Equations *(15)* and *(16)* agree if

$$\int_\chi = \frac{2}{3} K_0 - K_n \tag{17}$$

Even for such extreme variation from the ideal case where equation *(12)* holds as the Yolo light clay of the numerical example developed in parts 1 and 2 (12, 13), the left- and right-hand sides of equation *(17)* are at least of the same order of magnitude, being 4.7×10^{-6} cm. sec.$^{-1}$ and 8.1×10^{-6} cm. sec.$^{-1}$, respectively. Thus equation *(17)* should be very roughly true for all conceivable media and should give the order of magnitude of $\int_\chi$.

We therefore expect equation *(12)* to describe infiltration with a fair degree of accuracy throughout the full range of t. The error will be least for sandy and initially dry soils in which D will approach the ideal shape which makes equation *(12)* exact.

An equation based on the present analysis

Equation *(12)* is awkward to handle since t is the dependent variable. A simpler equation (11) with physical significance is

$$i = St^{1/2} + At \tag{18}$$

Obviously equation *(18)* is simply equation *(39)* of part 1 (12) with only the first two terms of the series retained and with

$$K_n + \int_\chi = A \tag{19}$$

The differential form of equation *(18)* is

$$v_0 = \tfrac{1}{2}St^{-1/2} + A \tag{20}$$

It is clear that as $t \to \infty$, equation *(18)* must fail, since for very large t

$$v_0 = K_0 \tag{21}$$

Equations *(20)* and *(21)* would agree if $A = K_0$, but this is not so. Semi-empirical methods of reconciling these equations at large t can be developed; these will generally involve at least one extra parameter K_0 and need not concern us here.

Equation (*18*), then, is a very simple equation, based on physical theory, which should be accurate for all but very large t. Its advantages are such that its use in applied hydrologic studies would seem desirable.

Tests of the various infiltration equations

Values of i at $t = 1000$ and $t = 10{,}000$ determined from our numerical example of infiltration into Yolo light clay [see parts 1 and 2 (12, 13)], were used as the basis of extrapolation as far as $t = 100{,}000$ by the four available infiltration equations. The Horton equation employs three parameters, so, in this instance, a third point ($t = 5500$) was also used. This test gave an indication of the reliability of the equations at small t. A second test was made using values at $t = 10{,}000$ and $t = 100{,}000$ as the basis of extrapolation to $t = 1{,}000{,}000$. (The value of i at $t = 55{,}000$ was also used for the Horton equation.) The second test gave a check on the reliability of the equations at large t. The results of these tests are summarized in table 1. Table 2, which compares the values of the pa-

TABLE 1

Extrapolation by four "infiltration equations"

Method of Computation	For $t = 10^5$ sec.		For $t = 10^6$ sec.	
	i (cm.)	% error	i (cm.)	% error
Value obtained from detailed analysis	4.477	0	18.670	0
Extrapolation by:				
Horton equation (*8*)	8.147	+82	29.412	+58
Kostiakov equation (*10*)	4.225	−5.6	15.395	−18
Equation (*12*)	4.448	−0.65	18.206	−2.6
Equation (*18*)	4.449	−0.63	17.753	−4.9

TABLE 2

Stability of parameters in four "infiltration equations"

Equation	Parameter	True Value	Value for Small-time Range	Value for Large-time Range
Horton (*8*)	v_i	—	572.9×10^{-6}	184.3×10^{-6}
	v_f	12.30×10^{-6}	76.06×10^{-6}	27.71×10^{-6}
	β	No physical significance	9.18×10^{-4}	0.92×10^{-4}
Kostiakov (*10*)	κ	No physical significance except at $t = 0$, where $\kappa = S$, $\alpha = \frac{1}{2}$	11.739×10^{-3}	9.311×10^{-3}
	α		0.5112	0.5634
Equation (*12*)	Y	142,900*	142,700	131,900
	Z	11.23*	11.23	10.32
Equation (*18*)	S	12.538×10^{-3}	12.534×10^{-3}	12.493×10^{-3}
	A	4.67×10^{-6}	4.85×10^{-6}	5.26×10^{-6}

* These values are derived from equation (*16*).

rameters determined for each time range and (where relevant) the true values, is also of interest.

Small percentage errors in table 1 indicate a good fit. So also does stability in the value of a parameter in table 2, and (where applicable) agreement with the true value available from the data of the problem or from the detailed analysis.

The bad failure of the Horton equation, despite its extra parameter, should be noted. The Kostiakov equation fits moderately well, especially at small t. Equations (*12*) and (*18*) fit very well indeed.

SUMMARY

A new physical property of porous media, *sorptivity*, is proposed. In some ways akin to permeability, this is essentially a measure of the capacity of the medium to absorb or desorb liquid by capillarity; the practical unit and dimensions of sorptivity are given. The *intrinsic sorptivity* (like intrinsic permeability) is an expression of the geometry of the medium, and is also defined and subjected to dimensional analysis.

Algebraic "infiltration equations" are discussed in the light of the analysis of the present series and are tested for goodness of fit in a numerical example. The Horton equation fails badly and the Kostiakov equation fits moderately well. An equation previously proposed by the author is found to be the exact outcome of the analysis for a special shape of D often approximated to in nature. That equation is awkward to handle, but a simpler infiltration equation derived from the analysis is also found to give good results, and seems well suited to the needs of applied hydrology. This equation is

$$i = St^{1/2} + At$$

where i is the cumulative infiltration, t is the time, S is the sorptivity, and A a second parameter which is also related to the analysis developed in this series.

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